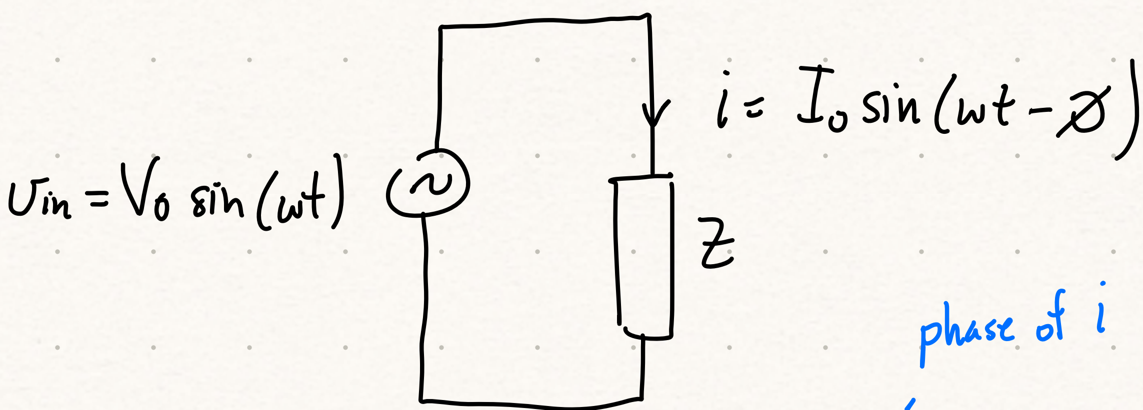
PHYS 231 Jake Bobowski Oct. 11, 2025

Analyze the frequency response of the series LRC circuit in two ways:

1. The easy way ⇒ using complex nos.
2. The hard way ⇒ no complex nos.

1. The Easy Way

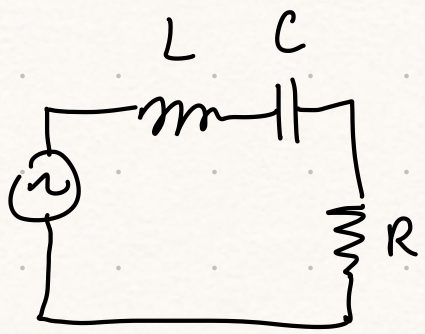
For <u>any</u> circuit of the form:



phase of i phase of Z

$$\boxed{I_0 = \frac{V_0}{|Z|} \qquad \tan\phi = \tan\phi_Z}$$

For the series LRC circuit:



$$Z = R + Z_L + Z_C$$
$$= R + j\omega L + \frac{1}{j\omega C}$$
$$= \underbrace{R}_{x} + j\underbrace{\left(\omega L - \frac{1}{\omega C}\right)}_{y}$$

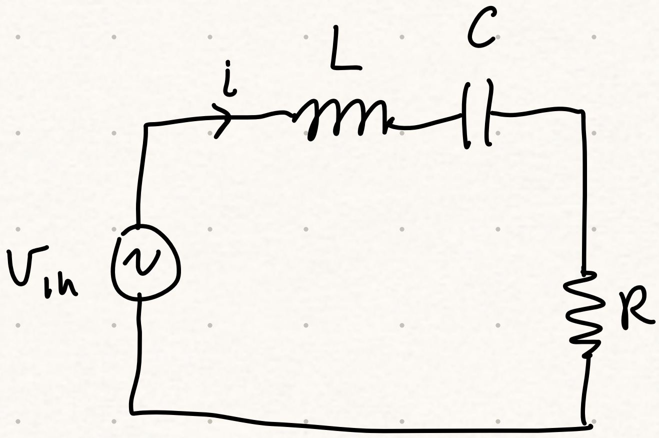
$$|Z| = \sqrt{x^2 + y^2}$$
$$= \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$
$$= R\sqrt{1 + \left(\omega\frac{L}{R} - \frac{1}{\omega RC}\right)^2}$$

$$\boxed{\therefore I_0 = \frac{V_0/R}{\sqrt{1 + \left(\omega\frac{L}{R} - \frac{1}{\omega RC}\right)^2}}}$$ #

$$\tan\phi_Z = \frac{y}{x} = \omega\frac{L}{R} - \frac{1}{\omega RC}$$

$$\boxed{\therefore \tan\phi = \omega\frac{L}{R} - \frac{1}{\omega RC}}$$ #

① The Hard Way



Kirchhoff loop rule

$$V_{in} - V_L - V_R - V_C = 0$$

$$\therefore V_{in} = L\frac{di}{dt} + Ri + \frac{1}{C}q$$

take another time derivative

$$\frac{dV_{in}}{dt} = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i$$

Take $V_{in} = V_0 \sin \omega t$ $\quad \frac{dV_{in}}{dt} = \omega V_0 \cos \omega t$

$$i = I_0 \sin(\omega t - \phi)$$

$$\frac{di}{dt} = \omega I_0 \cos(\omega t - \phi)$$

$$\frac{d^2i}{dt^2} = -\omega^2 I_0 \sin(\omega t - \phi)$$

$$\therefore \omega V_0 \cos\omega t = -\omega^2 I_0 L \sin(\omega t - \phi) + \omega I_0 R \cos(\omega t - \phi) + \frac{I_0}{C}\sin(\omega t - \phi)$$

$$\therefore \omega V_0 \cos\omega t = I_0\left(\frac{1}{C} - \omega^2 L\right)\sin(\omega t - \phi) + \omega I_0 R \cos(\omega t - \phi)$$

use the trig identities:

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\therefore \omega V_0 \underline{\cos\omega t} = \frac{I_0}{C}\left(1 - \omega^2 LC\right)\left[\underline{\sin\omega t}\cos\phi - \underline{\cos\omega t}\sin\phi\right] + \omega I_0 R\left[\underline{\cos\omega t}\cos\phi + \underline{\sin\omega t}\sin\phi\right]$$

collect $\sin\omega t$ & $\cos\omega t$ terms.

$$\underbrace{\left\{\omega V_0 + \frac{I_0}{C}\left(1-\omega^2 LC\right)\sin\phi - \omega I_0 R\cos\phi\right\}}_{1}\cos\omega t$$

$$= \underbrace{\left\{\frac{I_0}{C}\left(1-\omega^2 LC\right)\cos\phi + \omega I_0 R\sin\phi\right\}}_{2}\sin\omega t$$

$\cos\omega t$ & $\sin\omega t$ are orthogonal fcns. When $\cos\omega t = 0$, $\sin\omega t = \pm 1$. When $\sin\omega t = 0$, $\cos\omega t = \pm 1$.

$\therefore$ The only way the expression above can be valid $\forall\ t$ is if 1 and 2 are independently equal to zero.

Start w/ 2:

$$\frac{\cancel{I_0}}{C}\left(1-\omega^2 LC\right)\cos\phi + \omega \cancel{I_0} R\sin\phi = 0$$

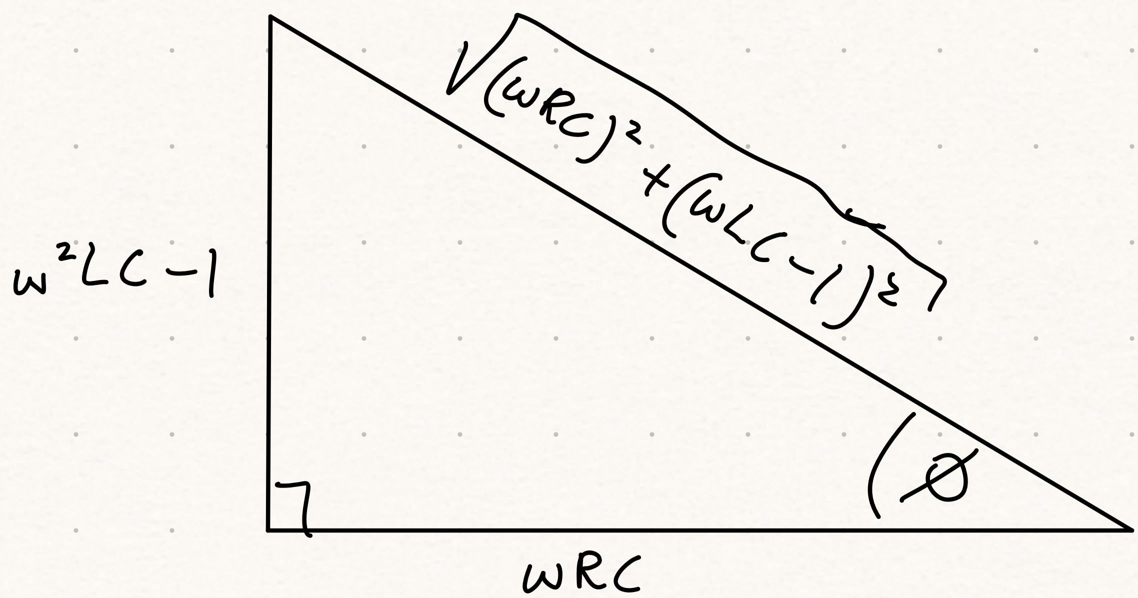
$$\therefore \left(\omega^2 LC - 1\right)\cos\phi = \omega RC\sin\phi$$

$$\therefore \tan\phi = \frac{\omega^2 LC - 1}{\omega RC} \quad (3)$$

or $$\boxed{\tan\phi = \omega\frac{L}{R} - \frac{1}{\omega RC}}$$ Agrees w/ (#)!

Before dealing w/ (1), we'll first use (#) to come up w/ some expressions for $\cos\phi$ & $\sin\phi$.

Use (3) to construct a right angle triangle



$$\tan\phi = \frac{\text{opp}}{\text{adj}} = \frac{\omega^2 LC - 1}{\omega RC}$$

$$\therefore \sin\phi = \frac{opp}{hyp} = \frac{\omega^2 LC - 1}{\sqrt{(\omega RC)^2 + (\omega LC - 1)^2}}$$

$$= \frac{\frac{\omega^2 LC - 1}{\omega RC}}{\sqrt{1 + \left(\frac{\omega LC - 1}{\omega RC}\right)^2}}$$

$$\boxed{\therefore \sin\phi = \frac{\omega\frac{L}{R} - \frac{1}{\omega RC}}{\sqrt{1 + \left(\omega\frac{L}{R} - \frac{1}{\omega RC}\right)^2}}} \quad (4)$$

$$\cos\phi = \frac{adj}{hyp} = \frac{\omega RC}{\sqrt{(\omega RC)^2 + (\omega LC - 1)^2}}$$

$$\boxed{\therefore \cos\phi = \frac{1}{\sqrt{1 + \left(\omega\frac{L}{R} - \frac{1}{\omega RC}\right)^2}}} \quad (5)$$

Want to use ① to solve for I_0.

$$\omega V_0 + \frac{I_0}{C}\left(1-\omega^2 LC\right)\sin\phi - \omega I_0 R\cos\phi = 0$$

divide by ωR

$$\frac{V_0}{R} + I_0\left[\left(\frac{1-\omega^2 LC}{\omega RC}\right)\sin\phi - \cos\phi\right] = 0$$

$$\therefore\ I_0 = \frac{V_0/R}{\cos\phi\left[1+\left(\frac{\omega L}{R}-\frac{1}{\omega RC}\right)\underbrace{\tan\phi}_{\frac{\omega L}{R}-\frac{1}{\omega RC}\ \text{from ④}}\right]}$$

$$\therefore\ I_0 = \frac{V_0/R}{\underbrace{\cos\phi}_{\frac{1}{\sqrt{1+\left(\frac{\omega L}{R}-\frac{1}{\omega RC}\right)^2}}\ \text{from ⑤}}\left[1+\left(\frac{\omega L}{R}-\frac{1}{\omega RC}\right)^2\right]}$$

$$\therefore \boxed{I_0 = \frac{V_0/R}{\sqrt{1+\left(\omega\frac{L}{R} - \frac{1}{\omega RC}\right)^2}}}$$

same as (#)!

These notes serve to demonstrate that our use of complex nos. in electronics is a matter of convenience, not necessity. However, it is worth emphasizing that the use of complex nos. greatly simplifies analysis and should, therefore, be embraced.

Finally, we note that, in Quantum mechanics complex nos. appear to be necessary. The most direct way of seeing why is to consider the full time-dependent Schrödinger equation, which can be thought of an axiom of quantum mechanics:

outside of electronics, $i \equiv \sqrt{-1}$ is typically used instead of j.

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t)$$

$V(x)$ has to be real. $\therefore$ If wavefunction $\Psi(x,t)$ was real, the entire RHS of Schrödinger's eq'n would be real making it impossible to be equal to the LHS of the equation.

$\therefore$ not only is Schrödinger's time-dependent equation complex, so too are its solutions $\Psi(x,t)$.